MA692: financial maths

Exercise sheet II (mostly adapted from Luenberger's book)

1. Suppose that a die is rolled twice and the average of two numbers is recorded as z. What is the mean and the variance of z? What happens to the variance as you roll the die n times and take the average of those n numbers as z?

2. Consider two assets A, B with correlation $\rho = 0.1$, $\bar{r}_A = 10\%$, $\bar{r}_B = 18\%$, $\sigma_A = 15\%$, $\sigma_B = 30\%$. Find the proportions α, $1 - \alpha$ to invest in A, B, such that the standard deviation of the resulting portfolio is minimised. Find the expected return and the standard deviation of the resulting portfolio.

3. Consider an investment of ₹1 million in staging a concert an year from now. There is 50% chance that this would return ₹3 million, but 50% chance that it would not take place at all and hence the return on investment will be zero (if it rains, or if the singer gets embroiled in a scandal, for example). The promoter can buy an failure insurance for the concert, a unit of which costs ₹0.50. The insurance pays ₹1 per unit if the concert does not take place and pays nothing if it does. Denoting u as the number of units purchased, what is the expected return on this investment? Show that the variance of this investment can be made zero by choosing an appropriate u, and find such u.

4. Consider 3 assets with variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$, mean returns $\bar{r}_1, \bar{r}_2, \bar{r}_3$ and with zero correlation with one another. Find the portfolio which achieves a mean return $\bar{r}$ and achieves minimum standard deviation for that mean return.

5. (adapted from Luenberger, Chapter 7): A company earns a rate of return r_A with variance σ_A^2 and has beta β_A. A fraction w of assets is owned by bondholders, and the remaining fraction $(1 - w)$ is owned by equity holders. Every year, the bondholders demand a riskless rate of return of r_B on their fraction of the assets, regardless of the actual rate of return r_A that was achieved that year. Beyond that, the equity holders take whatever is left after the bondholders have been paid. (Notice that there can only be one riskless rate of return, so that $r_B = r_{\mathrm{f}}$)

   (a) Why can there only be one riskless rate of return? Use an arbitrage argument.

   (b) What is the rate of return of the equity holders in terms of w and r_B?

   (c) What is the variance of this return in terms of σ_A^2 and w? As w grows (that is, the company becomes more leveraged) what happens to the variance?

   (d) What is the beta of the rate of return of the equity holders in terms of w and β_A?

6. Consider two investment opportunities: opportunity A has an expected payoff after 1 year a with mean ₹1.1 million, a standard deviation of ₹110,000 and a correlation with the market return of 0.4, whereas opportunity B has an expected payoff after 1 year with mean ₹1 million, a standard deviation of ₹100,000 and a correlation with the market return of -0.2. Both the opportunities require ₹1 million investment today.

The market portfolio return has mean 0.05 and a standard deviation 0.02. The risk-free rate is 0.01. Find the NPVs in both the cases and state which opportunity might be preferable.

7. A debt of ₹25,000 is to be amortized over 7 years with 7% interest per annum. What value of a constant monthly payment throughout 7 years would achieve this? How much is the total payment made over 7 years?

8. An investor has utility function $U(x) = x^{1/4}$ for salary. He has a new job offer which pays $\$80,000$ base salary with a bonus. the bonus will be $\$0$, $\$10,000$, $\$20,000$, $\$30,000$, $\$40,000$, $\$50,000$ or $\$60,000$, each with equal probability. What is the certainty equivalent of this salary structure?

9. Suppose that there is a risky project, whose possible outcomes at the end of the project period are as shown in the table below:

		Return	Probability
Project	High success	3.0	1/3
	Moderate success	2.0	1/3
	Failure	0.0	1/3
Risk free investment		1.5	1.0

There is also a risk-free investment, yielding 50% return over the period, as indicated in the table. Let (θ_1, θ_2) denote, respectively, the portfolio positions for the risky and risk-free assets. Assuming that the utility function takes the logarithmic form $U(x) = \ln(x)$, and also assuming the budget constraint $\theta_1 + \theta_2 = 1$, calculate the amount θ_1 to be invested in the risky project that maximises the expected utility.

10. (Adapted from Luenberger, chapter 3) Consider two bonds with annual payments as below:

End of year payment	Bond A	Bond B
year 1	100	0
year 2	100	0
year 3	$100 + 1000$	1000

Find prices of bonds if they are trading at 15% yield. Also find the corresponding duration in both the cases.

11. Consider an oil drilling venture with an initial investment of ₹875 and a payoff of ₹1000 after 1 year, with a volatility of 40%. Current risk-free rate is 10%, the rate of return on the market portfolio is 17% and the standard deviation of the market portfolio return is 12%. How does the venture compare with the capital market line?

12. Consider an investment which makes 10% return with 50% probability, 5% return with 30% probability and 1% return with 20% probability. Suppose that you are using a utility function $U(x) = \ln(2x)$. Find $\mathbb{E}(U(x))$, and hence find the certainty equivalent C such that $U(C) = \mathbb{E}(U(x))$.

13. Consider a cashflow $\{-b, b/2, b\}$, where $b > 0$ is a constant, and where the time period between two successive payments is assumed to be the same. Find the unique internal rate of return per period for this cash flow.

14. Consider a liability of ₹1000, payable after 5 years. Assume the yield curve to be flat at 5%. We consider two options for matching this liabilty: (1) purchase a 5 year zero coupon bond with a face value of ₹1000; (2) purchase a portfolio of two bonds - one 3 year zero coupon bond and one 7 year zero coupon bond, which match the duration and the present value of the liability. Find the portfolio composition in the second case. If the interest rate changes to 4.5% after the decision on liability matching, find the change in the present value of the two bond portfolio and in the present value of the single 5 year zero coupon bond.

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